

Toward an invariant definition of repulsive gravity

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A remarkable property of naked singularities in general relativity is their repulsive nature. The effects generated by repulsive gravity are usually investigated by analyzing the trajectories of test particles which move in the effective potential of a naked singularity. This method is, however, coordinate and observer dependent. We propose to use the properties of the Riemann tensor in order to establish in an invariant manner the regions where repulsive gravity plays a dominant role. In particular, we show that in the case of the Kerr-Newman singularity and its special subcases the method delivers plausible results.

1. Introduction

It is well known that the field equations of Einstein's theory of gravity allow the existence of exact solutions containing naked singularities. Moreover, recent studies indicate that under certain circumstances naked singularities can appear as the result of a realistic gravitational collapse.¹ An intriguing property of many naked singularities is that they can generate repulsive gravity. To understand this repulsive nature one can study the motion of test particles which, for example in the case of stationary axially symmetric fields, reduces to the study of an effective potential. Although the explicit form of the effective potential depends on the type of motion under consideration, in general one can find certain similitudes between the effective potential for geodesic motion and the effective Newtonian potential which follows from the metric as $g_{tt} \approx 1 - 2V_N = 1 - 2M_{eff}/r$, where the effective mass reduces to the physical mass M at infinity. One can then intuitively expect that in the regions where M_{eff} becomes negative, the effects of repulsive gravity may occur. In the case of the Schwarzschild metric the effective mass coincides with the physical mass, and repulsive gravity is obtained only if we change $M \rightarrow -M$; hence, the source of repulsion can be considered as unphysical. However, in the cases of the Reissner-Nordström and Kerr metrics we have respectively $M_{eff} = M - \frac{Q^2}{2r}$ and $M_{eff} = M - L(a, r, \theta)$,² leading to spacetime regions where repulsive gravity exists. The disadvantage of this approach is that it is clearly coordinate and observer dependent. The attempts to define gravitational repulsion in terms of curvature invariants³ and the behavior of light cones⁴ are also not definite. In this work we propose to use the eigenvalues of the curvature tensor to characterize repulsive

gravity in an invariant manner. We first consider the main second order curvature invariants and show that they do not reproduce the simple case of the Schwarzschild naked singularity. Then we show that the curvature eigenvalues provide a reasonable solution to the problem.

2. An invariant approach

From the curvature tensor one can form 14 functionally independent scalars of which only 4 are non-zero in empty space.⁵ As for the second order invariants, the most interesting are the Kretschmann scalar, $K_1 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, the Chern-Pontryagin scalar, $K_2 = [*R]_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, and the Euler scalar $K_3 = [*R^*]_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, where the asterisk represents dual conjugation. Although the use of these invariants has been proposed to define “repulsive domains” and negative effective masses in curved spacetimes,³ their quadratic structure does not allow to consider all possible cases of naked singularities. Indeed, for the Schwarzschild spacetime we get $K_1 = 48M^2/r^6$, whereas K_2 and K_3 are proportional to K_1 . Since the change $M \rightarrow -M$ does not affect the behavior of K_1 , these invariants do not recognize the presence of a Schwarzschild naked singularity. Similar difficulties appear in more general cases like the Kerr and Kerr-Newman naked singularities. Therefore, it seems necessary to consider the only first order invariant which is the curvature scalar R ; however, it vanishes identically in the empty space of naked singularities.

As an alternative approach we propose to use the eigenvalues of the curvature. To this end, consider the $SO(3, C)$ -representation of the curvature as follows. Let the line element be written in an (pseudo-)orthonormal frame as $ds^2 = \eta_{ab}\vartheta^a \otimes \vartheta^b$ with $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$. From the curvature 2-form $\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = \frac{1}{2}R^a_{bcd}\vartheta^c \wedge \vartheta^d$, where $d\vartheta^a = -\omega^a_b \wedge \vartheta^b$, one obtains the components of the curvature tensor whose irreducible parts are: the Weyl tensor, $W_{abcd} = R_{abcd} + 2\eta_{[a}[cR_{d][b]} + \frac{1}{6}R\eta_{a[d}\eta_{c]b}$, the trace-free Ricci tensor, $E_{abcd} = 2\eta_{[b}[cR_{d][a]} - \frac{1}{2}R\eta_{a[d}\eta_{c]b}$, and the curvature scalar, $S_{abcd} = -\frac{1}{6}R\eta_{a[d}\eta_{c]b}$, with $R_{ab} = \eta^{cd}R_{cabd}$. Furthermore, using the bivector notation for the indices $ab \rightarrow A$, according to $01 \rightarrow 1, 02 \rightarrow 2, 03 \rightarrow 3, 23 \rightarrow 4, 31 \rightarrow 5, 12 \rightarrow 6$, the curvature tensor can be written as $R_{AB} = W_{AB} + E_{AB} + S_{AB}$ with

$$W_{AB} = \begin{pmatrix} N & M \\ M & -N \end{pmatrix}, E_{AB} = \begin{pmatrix} P & Q \\ Q & -P \end{pmatrix}, S_{AB} = \frac{R}{12} \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}. \quad (1)$$

Here M , N and P are (3×3) real symmetric matrices, whereas Q is antisymmetric. The $SO(3, C)$ -representation corresponds to $\mathbf{R} = W + E + S$ with $W = M + iN$, $E = P + iQ$, and $S = \frac{1}{12}RI_3$ (see⁶ for more details.) The eigenvalues of the curvature matrix \mathbf{R} are in general complex $\lambda_n = a_n + ib_n$ and, according to Petrov’s classification,⁷ are an invariant characterization of the curvature tensor. Moreover, in the most general case of gravitational fields belonging to Petrov’s class I, we obtain the largest number of eigenvalues, namely $n = 3$.

In the special case of the Schwarzschild metric there is only one eigenvalue $\lambda = M/r^3$ and the change $M \rightarrow -M$ induces a drastic change in the eigenvalue and

in the structure of spacetime as well. An analysis of the more general Kerr-Newman naked singularity indicates that in fact the curvature eigenvalues change their sign and present several maxima and minima in the vicinity of the singularity which is exactly the region where repulsive gravity appears. It then seems reasonable to introduce the concept of region of repulsion as the region of spacetime contained between the first extremum of the eigenvalue, when approaching from spatial infinity, and the singularity. The extremum is defined in an invariant manner as $\partial\lambda_n/\partial x^i = 0$, where x^i are the spatial coordinates. This invariant approach leads to the following values for the Reissner-Nordström and Kerr naked singularities

$$R_{rep}^{RN} = 2\frac{Q^2}{M}, \quad R_{rep}^K = (1 + \sqrt{2})a \cos \theta, \quad (2)$$

respectively. These results are in agreement with the analysis of test particles. In fact, the Reissner-Nordström singularity presents repulsion effects outside the classical radius $R_{class} = Q^2/M$, and the radius of repulsion $R_{rep}^{RN} = 2R_{class}$ is always situated within the zone of instability of circular motion. The Kerr naked singularity turns out to be attractive only on the equatorial plane [$R_{rep}^K(\pi/2) = 0$], and it is repulsive otherwise. The case of the Kerr-Newman singularity cannot be solved analytically in a compact form. On the axis, however, the radius of repulsion is given by the largest root of the equation $Mr^4 - 2Q^2r^3 - 6Ma^2r^2 + 2a^2Q^2r + Ma^4 = 0$. Introducing values for the mass, charge and angular momentum the resulting radius of repulsion is always situated in the region where the motion of test particles is affected by repulsive gravity.

Our invariant approach to define repulsive gravity leads to plausible and physical reasonable results in the case of naked singularities which possess a black hole counterpart. The investigation of naked singularities generated by a mass quadrupole moment (without black hole counterparts) indicates that our method consistently delivers the expected results. Moreover, it turns out that the concept of region of repulsion can be used as a criterion to study the problem of matching interior and exterior solutions of Einstein's equations.

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